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Flat connections on punctured surfaces and geodesic polygons in a Lie group

Indranil Biswas *, Saikat Chatterjee

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract

Let S be a subset of n points on a compact connected oriented surface M of genus g, and let G be a compact semisimple Lie group. The space of isomorphism classes of flat G-connections on $P := M \setminus S$ with fixed conjugacy class of monodromy around each point of S will be denoted by $\mathcal{R}$. It is known that $\mathcal{R}$ has a natural symplectic structure. We relate $\mathcal{R}$ with the space of geodesic $(4g+n)$-gons in G. A natural 2-form on the space of geodesic $(4g+n)$-gons is constructed using the Killing form on $\mathrm{Lie}(G)$. We establish an identity between the symplectic form on $\mathcal{R}$ and this 2-form on geodesic $(4g+n)$-gons in G.

MSC: 53D30; 53C05

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1. introduction

Let M be a compact connected oriented surface of genus g. Fix a finite subset

$$S := \{s_1, s_2, \dots, s_n\} \subset M.$$

The complement $M \setminus S$ will be denoted by P. Fix a compact connected semisimple Lie group G. By a conjugacy class in G we will mean an orbit for the adjoint action of G on itself. For each point $s_i \in S$, fix a conjugacy class $C_{s_i} \subset G$. The adjoint action of G on itself produces an action of G on the space of homomorphisms from $\pi_1(P)$ to G. Let

* Corresponding author.
E-mail addresses: indranil@math.tifr.res.in (I. Biswas), saikat@math.tifr.res.in (S. Chatterjee).

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$$\mathcal{R} \subset \operatorname{Hom}\bigl(\pi_1(P), G\bigr)/G$$

be the subspace consisting of all homomorphisms ρ such that ρ takes the oriented loop around s_i into C_{s_i} for every i. This $\mathcal{R}$ has a natural symplectic structure (see [3–6,1]).

Let B be the Killing form on $\mathfrak{g} := \operatorname{Lie}(G)$. A Riemannian metric on G is constructed by taking G-translations of B. A geodesic N-gon in G is a continuous mapping

$$\tau : [1, N] \longrightarrow G$$

such that $\tau(0) = \tau(N) = e$ and for each integer $1 \leqslant i \leqslant N$, the restriction of τ to $[i-1, i]$ is a geodesic in G. Let $\mathfrak{D}(N)$ denote the space of all geodesic N-gons in G. There is a natural C^∞ two-form on $\mathfrak{D}(N)$ defined by the trilinear form $(a, b, c) \longmapsto B(a, [b, c])$ on $\mathfrak{g}$. This two-form on $\mathfrak{D}(N)$ will be denoted by θ.

Our aim here is to establish a relationship between this two-form θ on $\mathfrak{D}(4g+n)$ and the symplectic form on $\mathcal{R}$. This was done earlier in [2] for the special of $g = 0$. We recall this result of [2].

Take $M = S^2$. Let $\widetilde{\Omega}_0$ be the symplectic form on $\mathcal{R}$. For each $i \in n$, fix a conjugacy class $\widetilde{C}_{s_i} \subset \mathfrak{g}$ for the adjoint action of G on $\mathfrak{g}$. Assume that $\exp(\widetilde{C}_{s_i}) = C_{s_i}$, where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map.

Let $\widetilde{A}$ be the space of all maps f from S to $\mathfrak{g}$ such that $f(s_i) \in \widetilde{C}_{s_i}$ and

$$\prod_{j=1}^{n} \exp f(s_j) = e,$$

where $e \in G$ is the identity element. The group G acts on $\widetilde{A}$ through conjugation. Let

$$\widetilde{\psi} : \widetilde{A} \longrightarrow \widehat{A} := \widetilde{A}/G$$

be the quotient map for the action of G. We have a natural map

$$\widetilde{H} : \widehat{A} \longrightarrow \mathcal{R}$$

constructed using the exponential map from $\mathfrak{g}$ to G.

Let

$$\widetilde{\phi} : \widetilde{A} \longrightarrow \mathfrak{D}(n)$$

be the map that sends any $f \in \widetilde{A}$ to the geodesic n-gon on G whose i-th edge is

$$t \longmapsto \exp f(s_1) \exp f(s_2) \cdots \exp t f(s_i), \quad t \in [0, 1].$$

In Theorem 1.1 of [2] it was shown that the two-form $\widetilde{\phi}^*\theta$ on $\widetilde{A}$ coincides with the pullback, by the map $\widetilde{H} \circ \widetilde{\psi}$, of the symplectic form on $\mathcal{R}$.

In Theorem 2.1, the above theorem of [2] is generalized to surfaces of arbitrary genus.

2. Symplectic form and geodesic polygons

Let M be a compact connected oriented surface of genus g. We fix n points $S := \{s_1, s_2, \ldots, s_n\}$ of M. Let $P := M \setminus S$ be the punctured surface. Fix a base point $p \in P$. The fundamental group $\pi = \pi_1(P, p)$ is generated by $2g + n$ elements

$$\Gamma := \{a_j, b_j, m_i \mid 1 \leqslant j \leqslant g,\ 1 \leqslant i \leqslant n\}$$

admitting a single relation

$$\left(\prod_{i=1}^{g} a_j b_j a_j^{-1} b_j^{-1}\right) \prod_{j=1}^{n} m_i = 1.$$

Let G be a connected compact semisimple Lie group; its Lie algebra will be denoted by $\mathfrak{g}$. Fix a conjugacy class $\widetilde{C}_{s_i} \subset \mathfrak{g}$ for each $i \in [1, n]$. Let A be the space of all maps

$$f : \Gamma \longrightarrow \mathfrak{g}$$

such that

$$f(m_i) \in \widetilde{C}_{s_i}, \qquad f\big(a_j^{-1}\big) = -f(a_j), \qquad f\big(b_j^{-1}\big) = -f(b_j), \tag{2.1}$$

and the following condition holds:

$$\prod_{i=1}^{g} \exp\big(f(a_i)\big) \exp\big(f(b_i)\big) \exp\big(-f(a_i)\big) \exp\big(-f(b_i)\big) \prod_{j=1}^{n} \exp\big(f(m_j)\big) = e. \tag{2.2}$$

We will construct an action of G on A. For any $h \in G$ and $f \in A$, define

$$(h \circ f)(a_j) := \mathrm{Ad}_h\big(f(a_j)\big), \qquad (h \circ f)(b_j) := \mathrm{Ad}_h\big(f(b_j)\big),$$
$$(h \circ f)(m_i) := \mathrm{Ad}_h\big(f(m_i)\big).$$

Note that the condition in (2.2) is preserved by this action of G. Thus writing explicitly,

$$\prod_{i=1}^{g} \exp\big((h \circ f)(a_i)\big) \exp\big((h \circ f)(b_i)\big) \exp\big(-(h \circ f)(a_i)\big)$$
$$\times \exp\big(-(h \circ f)(b_i)\big) \prod_{j=1}^{n} \exp(h \circ f)(m_j) = e.$$

This produces an action of G on A where any $h \in G$ acts as $f \longmapsto h \circ f$. Define the quotient space for this action of G on A

$$W := \frac{A}{G}$$

and the quotient map

$$\psi : A \longrightarrow W. \tag{2.3}$$

For any given $f \in A$, by sending each $a_j, b_j, m_i \in \Gamma$ to $\exp(f(a_j))$, $\exp(f(b_j))$, $\exp(f(m_i))$ respectively, we obtain a homomorphism from Γ to G. Define

$$\mathcal{R} := \frac{\{\rho \in \mathrm{Hom}(\Gamma, G) \mid \rho(m_i) \in C_{s_i}\}}{G} \subset \frac{\mathrm{Hom}(\Gamma, G)}{G},$$

where $C_{s_i} = \exp(\widetilde{C}_{s_i})$. Let

$$H : W \longrightarrow \mathcal{R} \tag{2.4}$$

be the map defined by

$$H(f)(a_j) := \exp\bigl(f(a_j)\bigr), \qquad H(f)(b_j) := \exp\bigl(f(b_j)\bigr),$$
$$H(f)(m_i) := \exp\bigl(f(m_i)\bigr) \in C_{s_i},$$

where $f \in W$.

Let B be the G-invariant symmetric bilinear 2-form on $\mathfrak{g}$ defined by the Killing form. Let $E \longrightarrow P$ be a flat principal G-bundle. Let

$$\iota : P \hookrightarrow M$$

be the inclusion map. The G-invariant bilinear form B induces a bilinear form on the adjoint vector bundle $\mathrm{ad}(E)$ of E. The connection on E induces a connection on $\mathrm{ad}(E)$. Let $\Sigma(\mathrm{ad}(E))$ denote the sheaf of locally defined flat sections of $\mathrm{ad}(E)$. Define the direct image $\iota_*\Sigma(\mathrm{ad}(E)) \longrightarrow M$.

Let ω be the space of all flat connections on E. The space $\mathcal{R}$ is the space of all isomorphism classes of (E, ∇), where $\nabla \in \omega$ satisfying the condition that for any $s_i \in S$ and any loop c_i around s_i,

$$\exp \int_{c_i} \nabla \in C_{s_i}$$

(C_{s_i} is the fixed conjugacy class associated to s_i). The tangent space at any $\rho \in \mathcal{R}$ is given by

$$T_\rho \mathcal{R} = H^1\bigl(M, \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr)\bigr)$$

(see [3]). Hence

$$T_\rho \mathcal{R} \otimes T_\rho \mathcal{R} = H^1\bigl(M, \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr)\bigr) \otimes H^1\bigl(M, \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr)\bigr) \longrightarrow H^2\bigl(M, \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr) \otimes \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr)\bigr).$$

As B is G-invariant, we have a bilinear pairing

$$\widetilde{B} : \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr) \otimes \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr) \longrightarrow \mathbb{R},$$

and hence

$$H^2\bigl(M, \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr) \otimes \iota_*\Sigma\bigl(\mathrm{ad}(E)\bigr)\bigr) \xrightarrow{\widetilde{B}_*} H^2(M, \mathbb{R}) = \mathbb{R}.$$

The natural symplectic form Ω_0 on $\mathcal{R}$ is given by the homomorphism

$$T_\rho \mathcal{R} \otimes T_\rho \mathcal{R} \longrightarrow \mathbb{R}$$

obtained by combining the above two homomorphisms [3,1]. We define the form

$$\Omega := H^*\Omega_0$$

on W, where H is constructed in (2.4).

Next we describe the tangent space of A. Let us define for any a_j, b_j or m_i, and $f \in A$,

$$\mathfrak{g}\bigl(f(a_j)\bigr) := \bigl[f(a_j), \mathfrak{g}\bigr] \subset \mathfrak{g}, \qquad \mathfrak{g}\bigl(f(b_j)\bigr) := \bigl[f(b_j), \mathfrak{g}\bigr] \subset \mathfrak{g},$$
$$\mathfrak{g}\bigl(f(m_i)\bigr) := \bigl[f(m_i), \mathfrak{g}\bigr] \subset \mathfrak{g},$$

where $1 \leqslant j \leqslant g$ and $1 \leqslant i \leqslant n$. Let V_f be the space of all functions $v : \Gamma \longrightarrow \mathfrak{g}$ satisfying the following conditions:

(1) $v(a_j) \in \mathfrak{g}(f(a_j))$,
(2) $v(b_j) \in \mathfrak{g}(f(b_j))$,
(3) $v(m_i) \in \mathfrak{g}(f(m_i))$,
(4) $\sum_{\mu=1}^{4g+n} \mathrm{Ad}_{f_\mu}(v(\alpha_\mu)) = 0$,

where f_μ is defined as follows:

$$\begin{aligned}
f_1 &= \exp f(a_1),\\
f_2 &= \exp f(a_1) \exp f(b_1),\\
f_3 &= \exp f(a_1) \exp f(b_1) \exp\bigl(-f(a_1)\bigr),\\
f_4 &= \exp f(a_1) \exp f(b_1) \exp\bigl(-f(a_1)\bigr) \exp\bigl(-f(b_1)\bigr),\\
&\vdots\\
f_{4g} &= \prod_{j=1}^{g} \exp f(a_j) \exp f(b_j) \exp\bigl(-f(a_j)\bigr) \exp\bigl(-f(b_j)\bigr),\\
f_{4g+k} &= f_{4g} \prod_{i=1}^{n} \exp f(m_i), \quad 1 \leqslant k \leqslant n, \qquad (2.5)
\end{aligned}$$

and $\alpha_1 = a_1, \alpha_2 = b_1, \alpha_3 = a_1^{-1}, \alpha_4 = b_1^{-1}, \alpha_5 = a_2, \ldots, \alpha_{4g} = b_g^{-1}, \alpha_{4g+i} = m_i, 1 \leqslant i \leqslant n$.

Proposition 2.1. *The tangent space $T_f A$ is identified with V_f defined above.*

Proof. As $\mathfrak{g}$ is a Lie algebra, the conditions (1) and (2) are obtained. Also from (2.1) we know that $f(m_i)$ is in the conjugacy class C_{s_i}, hence under the Lie bracket it remains in C_{s_i}; so (2.1) leads to condition (3). On the other hand, (2.2) is preserved under the adjoint action, hence (2.2) leads to the condition (4). Consequently, V_f gets identified with $T_f A$. □

In view of Proposition 2.1, we have the following identity (see [1, (3.5)] and [1, Theorem 1]):

Proposition 2.2.

$$\sum_{1}^{n+4g} B\bigl(f(\alpha_\mu), \bigl[v(\alpha_\mu), w(\alpha_\mu)\bigr]\bigr) = \bigl(\psi^* H^* \Omega_0\bigr)(v, w), \qquad (2.6)$$

where $\alpha_1 = a_1$, $\alpha_2 = b_1$, $\alpha_3 = a_1^{-1}$, $\alpha_4 = b_1^{-1}$, $\alpha_5 = a_2$, $\ldots$, $\alpha_{4g+i} = m_i$, $1 \leqslant i \leqslant n$, Ω_0 is the canonical symplectic form on $\mathcal{R}$ and $v, w \in V_f$.

For $g = 0$, this was proved in [2, Proposition 2.2].

Let us construct a $(4g + n)$-sided geodesic polygon from a given $f \in W$. Vertices of the polygon are:

$$\begin{aligned}
f_1 &= \exp\bigl(f(a_1)\bigr),\\
f_2 &= \exp\bigl(f(a_1)\bigr) \exp\bigl(f(b_1)\bigr),\\
f_3 &= \exp\bigl(f(a_1)\bigr) \exp\bigl(f(b_1)\bigr) \exp\bigl(-f(a_1)\bigr),
\end{aligned}$$

$$
\begin{aligned}
f_4 &= \exp\bigl(f(a_1)\bigr)\exp\bigl(f(b_1)\bigr)\exp\bigl(-f(a_1)\bigr)\exp\bigl(-f(b_1)\bigr),\\
&\ \vdots\\
f_{4g} &= \prod_{j=1}^{g}\exp\bigl(f(a_j)\bigr)\exp\bigl(f(b_j)\bigr)\exp\bigl(-f(a_j)\bigr)\exp\bigl(-f(b_j)\bigr),\\
f_{4g+k} &= f_{4g}\prod_{i=1}^{n}\exp\bigl(f(m_i)\bigr), \quad 1 \leqslant k \leqslant n.
\end{aligned}
\tag{2.7}
$$

So from (2.2) we have

$$f_{4g+n} = e.$$

Following the construction of [2], we set $f_0 = e$. For any $1 \leqslant \mu \leqslant 4g+n$, the edge that connects the vertices $f_{\mu-1}$ and f_μ will be denoted by $l^{\mu}_{\mu-1}$. Each edge is the geodesic segment defined as follows:

$$
\begin{aligned}
&l^1_0 : t \longmapsto \exp\bigl(tf(a_1)\bigr),\\
&l^2_1 : t \longmapsto f_1\exp\bigl(tf(b_1)\bigr),\\
&l^3_2 : t \longmapsto f_2\exp\bigl(-tf(a_1)\bigr),\\
&l^4_3 : t \longmapsto f_3\exp\bigl(-tf(b_1)\bigr),\\
&l^5_4 : t \longmapsto f_4\exp\bigl(tf(a_2)\bigr),\\
&\quad \vdots\\
&l^{4g}_{4g-1} : t \longmapsto f_{4g-1}\exp\bigl(-tf(b_g)\bigr),\\
&l^{4g+k}_{4g+k-1} : t \longmapsto f_{4g+k-1}\exp\bigl(tf(m_k)\bigr), \quad 1 \leqslant k \leqslant n,
\end{aligned}
\tag{2.8}
$$

where $t \in [0,1]$.

Let us denote the space of all such geodesic N-gons by $\mathfrak{D}(N)$. The polygon described above in (2.7) and (2.8) is a geodesic $(4g+n)$-gon in G. So we have a map

$$\phi : W \longrightarrow \mathfrak{D}(4g+n) \tag{2.9}$$

that sends any $f \in W$ to the polygon described in (2.8).

We recall the construction of the 2-form θ on $\mathfrak{D}(N)$ (see [2]). Let θ_0 is the trilinear form on $\mathfrak{g}$ given by

$$\theta_0 : \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathbb{R}; \quad (a,b,c) \longmapsto B\bigl(a,[b,c]\bigr). \tag{2.10}$$

As B is G-invariant, we get a 3-form $\bar{\theta}$ on G using translations

$$\bigl(g'.\theta_0\bigr) = \bar{\theta}\bigl(g'\bigr), \tag{2.11}$$

where $g' \in G$. Let τ be a geodesic N-gon on G. A tangent vector $v \in T_\tau\mathfrak{D}(N)$ is given by Jacobi fields along each l^i_{i-1}, where l^i_{i-1} is the edge (geodesic line segment) joining vertices $i-1$ and i. Take $v, w \in T_\tau\mathfrak{D}(N)$; let $\bar{v}$ and $\bar{w}$ be the corresponding sections of $\tau^*T\mathfrak{D}(N)$. Now $\tau^*\bar{\theta}(v,w)$ gives a piece-wise smooth 1-form on $[1,N]$. Denoting integral of this 1-form on $[1,N]$ by $\theta(v,w) \in \mathbb{R}$, we have a 2-form θ on $\mathfrak{D}(N)$, which is given by

$$\theta : (v, w) \longmapsto \theta(v, w) \tag{2.12}$$

for any $v, w \in T_\tau \mathfrak{D}(N)$.

Let $g' \in G$ and $\alpha \in \mathfrak{g}$. Consider the geodesic segment

$$l : [0, 1] \longrightarrow G$$

defined by $t \longmapsto g' \exp(t\alpha)$. For any $\beta, \gamma \in \mathfrak{g}$, let $\overline{\beta}$ and $\overline{\gamma}$ be the corresponding Jacobi field along l. Contracting $\overline{\theta}$ (constructed in (2.11)) by $\overline{\beta}$ and $\overline{\gamma}$ we obtain a 1-form on [0, 1]. Denote this 1-form by $\sigma(\beta, \gamma)$. We have

$$\int_0^1 \sigma(\beta, \gamma) = \theta_0(\alpha, \beta, \gamma) \tag{2.13}$$

(see [2, Proposition 2.3]); so, from the definition of θ_0 in (2.10),

$$\int_0^1 \sigma(\beta, \gamma) = B\bigl(\alpha, [\beta, \gamma]\bigr).$$

Let τ be a geodesic $(4g + n)$-gon as described in (2.8) and (2.7). For each edge $l^{\mu}_{\mu-1}$, $1 \leqslant \mu \leqslant 4g + n$, we apply (2.13); from Proposition 2.2,

$$\phi^* \theta|_{l^{\mu}_{\mu-1}} = \psi^* H^* \Omega_0|_{l^{\mu}_{\mu-1}},$$

where ϕ is defined in (2.9). Thus we have proven:

Theorem 2.1. *For a surface of genus g with n punctures, if the symplectic form on $\mathcal{R}$ is Ω_0 and ϕ, ψ, H and θ are defined as in* (2.9)*,* (2.3)*,* (2.4) *and* (2.12) *respectively, then*

$$\psi^* H^* \Omega_0 = \phi^* \theta.$$

References

[1] A.Y. Alekseev, A.Z. Malkin, Symplectic structure of the moduli space of flat connections on a Riemann surface, Comm. Math. Phys. 169 (1995) 99–119.

[2] I. Biswas, Flat connections on a punctured sphere and geodesic polygons in a Lie group, J. Geom. Phys. 39 (2001) 129–134.

[3] I. Biswas, K. Guruprasad, Principal bundles on open surfaces and invariant functions on Lie groups, Internat. J. Math. 4 (1993) 535–544.

[4] W. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984) 200–225.

[5] K. Guruprasad, Flat connections, geometric invariants and the symplectic nature of the fundamental group of surfaces, Pacific J. Math. 162 (1994) 45–55.

[6] K. Guruprasad, J. Huebschmann, L. Jeffrey, A. Weinstein, Group systems, groupoids, and moduli spaces of parabolic bundles, Duke Math. J. 89 (1997) 377–412.